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ON THE DIFFERENTIATION OF DEFINITE INTEGRALS

BY WM. F. OSGOOD

THE object of this paper is to give a simpler proof of the theorem that

$$(A) \quad \frac{d}{da} \int_a^b f(x, a) dx = \int_a^b \frac{\partial f}{\partial a} dx + f(b, a) \frac{db}{da} - f(a, a) \frac{da}{da}$$

than those that are current.

1. **The Common Proofs.** The theorem is usually proven by writing

$$\phi(a) = \int_a^b f(x, a) dx,$$

forming the difference :

$$\begin{aligned} \phi(a + \Delta a) - \phi(a) &= \int_{a + \Delta a}^{b + \Delta a} f(x, a + \Delta a) dx - \int_a^b f(x, a) dx \\ &= \left(\int_{a + \Delta a}^a + \int_a^b + \int_b^{b + \Delta a} \right) f(x, a + \Delta a) dx - \int_a^b f(x, a) dx \\ &= \int_a^b [f(x, a + \Delta a) - f(x, a)] dx + \int_b^{b + \Delta a} f(x, a + \Delta a) dx \\ &\quad - \int_a^{a + \Delta a} f(x, a + \Delta a) dx, \end{aligned}$$

and applying the theorems of mean value to these last three integrals ; suitable assumptions being made about the continuity of the functions that enter. Cf., for example, Goursat-Hedrick, *Mathematical Analysis*, vol. I, §97.

Another proof consists in changing the variable of integration so that the new limits of integration become constant :

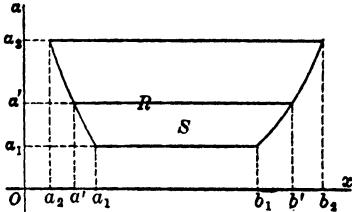
$$t = \frac{x - a}{b - a}, \quad x = (b - a)t + a,$$

$$\int_a^b f(x, a) dx = (b - a) \int_0^1 f(x, a) dt,$$

(119)

this latter case having been treated previously. Cf. Picard, *Traité d'analyse*, vol. I, chap. I, §18.

2. Critique of these Proofs. We will first state the theorem in detail in its simplest and most useful form. Let R be a region of the (x, a) -plane bounded by the right lines



$$a = a_1, \quad a = a_2, \quad (a_1 < a_2),$$

and the curves

$$a = \psi(x), \quad b = \omega(x),$$

where each of the functions $\psi(x)$ and $\omega(x)$ shall be continuous, together with its first derivative, throughout the interval $a_1 \leq a \leq a_2$, and where

$$\psi(x) < \omega(x).$$

The region R shall include its boundary.

The function $f(x, a)$ shall be continuous at all points of R . Its partial derivative

$$\frac{\partial f}{\partial a} = f_a(x, a)$$

shall exist and be continuous at all interior points of R , and this function, $f_a(x, a)$, shall remain finite throughout the interior of R .

Under the above conditions the integral

$$\int_a^b f(x, a) dx$$

defines a function of a having a derivative given by the above formula, (A).

Turning now to the first of the proofs cited in §1 we see that if, in the neighborhood of a point $a = a'$ of the interval $a_1 \leq a \leq a_2$, the functions $\psi(a)$ and $\omega(a)$ are both monotonic, the above transformation of the difference of the integrals, or a suitable modification of this transformation, is legitimate and the proof is sound. But there are several cases to consider when Δa and Δb change sign with Δa .

If, in particular, the functions $f(x, a)$, $f_a(x, a)$ are continuous throughout a larger region R' lying between the same parallels $a = a_1$ and $a = a_2$ and containing all the interior and boundary points of R situated between these

lines in its interior, the single transformation of the difference given above will hold for both positive and negative values of Δa . But the theorem thus restricted is too narrow for all the ordinary applications of practice.

There are, then, even when $\psi(a)$ and $\omega(a)$ are both monotonic, several cases of the transformation of the difference of the above integrals to be considered.

If, however, one of the functions $\psi(a)$, $\omega(a)$ is not monotonic; for example, if

$$\begin{aligned}\omega(a) &= (a - a')^3 \sin \frac{1}{a - a'}, & a \neq a', \\ \omega(a') &= 0,\end{aligned}$$

there is trouble. The theorem is still true, but a special ϵ -proof is necessary.

The second proof is not open to this objection, but it turns out that it is necessary to assume the existence of the partial derivative of $f(x, a)$ with respect to x , and thus the proof does not apply to the theorem stated in the above generality.*

3. A New Proof. We can obtain a simple proof of the theorem as follows. Consider the double integral

$$(1) \quad I = \iint_S f_a(x, a) dS,$$

extended over the region

$$S: \quad a \leq x \leq b, \quad a_1 \leq a \leq a',$$

when $a_1 < a' \leq a_2$. We can first evaluate I by means of the iterated integral

$$(2) \quad I = \int_{a_1}^{a'} da \int_a^b f_a(x, a) dx.$$

Secondly, we can evaluate I by means of Green's Theorem:

$$\iint_S f_a(x, a) dS = - \int_C f(x, a) dx.$$

* By means of this transformation, however, a simple proof can be given that

$$\int_a^b f(x, a) dx$$

is a continuous function of a if $f(x, a)$ is continuous in R , and a, b are continuous functions of a . The existence of a derivative with respect to x is here unnecessary.

Hence

$$(3) \quad I = - \int_{a_1}^{b_1} f(x, a_1) dx - \int_{a_1}^{a'} f(b, a) \frac{db}{da} da \\ + \int_{a'}^{b'} f(x, a') dx + \int_{a_1}^{a'} f(a, a) \frac{da}{da} da.$$

Equating the two expressions (2) and (3) for I , dropping the accent against the a , and differentiating the equation thus resulting with respect to a , we obtain the theorem contained in the formula (A).

Instead of using the region S between the parallels $a = a_1$ and $a = a'$, we might have chosen an arbitrary value $a_3 \neq a'$. The two evaluations of I would have yielded the same final equation, the subscript 1 being merely replaced by 3. Thus, in particular, if $a_3 = a_2$, the excluded value $a' = a_1$ no longer presents an exception.

4. Generalizations. Corresponding to more general forms of Green's Theorem we obtain the theorem under consideration with less restrictive hypotheses. Thus if $f_a(x, a)$ is continuous within R , but does not remain finite on the boundary, the function $f(x, a)$ still being assumed continuous in R , and if the surface integral (1) converges when extended over any region

$$S': \quad a \leq x \leq b, \quad a_1 \leq a \leq a_3,$$

where $a_1 < a_1 < a_2 < a_2$, and if, moreover, the first integral to be evaluated in (2), namely:

$$\int_a^b f_a(x, a) dx,$$

converges uniformly in the interval $a_1 \leq a \leq a_2$, the above proof will hold for all values of a' such that $a_1 < a' < a_2$.

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